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To cite this article: Sergey Novokshonov 2019 *J. Phys.: Conf. Ser.* **1389** 012104

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Quantization of the anomalous Hall conductance in a disordered magnetic Chern insulator

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Abstract. The intrinsic anomalous Hall conductance σ_{xy}^{int} of a minimal model of the two-dimensional disordered Chern insulator is investigated in the framework of Kubo quantum theory of linear response. The electron momentum relaxation is assumed to be due to electron scattering by Gaussian white-noise potential. The explicit expressions for the density of states and σ_{xy}^{int} are obtained in the self-consistent Born approximation. The numerical analysis of these expressions at the different values of parameters of the considered model shows that calculated σ_{xy}^{int} takes a quantized value $e^2/4\pi\hbar$ when the Fermi level lies within the energy gap. This gap is narrowed as disorder increases that leads to decreasing of the Hall plateau width.

1. Introduction

In the last years, a widespread interest was attracted to investigation of the quantum anomalous Hall effect (QAHE) in the topological nontrivial materials [1–3]. The possibility of quantizing Hall conductance without Landau levels was first demonstrated by Haldane [4] based on a tight-binding model with a zero net magnetic flux through the unit cell. Subsequently, several authors predicted the quantization of anomalous Hall conductance in two-dimensional magnets with spin-orbit interaction [5–8]. This effect was first discovered experimentally by Chang with co-workers [9] in the thin films of the bismuth based topological insulator doped by Cr.

According to [5] the anomalous Hall conductance σ_{xy} of a two-dimensional magnetic topological insulator takes quantized value, when the Fermi level lies within the gap between the bulk energy bands. In this regime, the proposed by Karplus and Luttinger [10] intrinsic mechanism contributes to σ_{xy} . This mechanism is caused by nontrivial topology of the electron states in ideal crystal [6, 7], as result the quantized value of the Hall conductance is proportional to the topological Chern number Ch [11] $\sigma_{xy} = \sigma_0 \text{Ch}$, where $\sigma_0 = e^2/2\pi\hbar$ is the conductance quantum. For the two-band Hamiltonian $\mathcal{H} = \mathbf{d}_p \cdot \boldsymbol{\sigma}$ of a two-dimensional system (2), the Chern number is expressed as

$$\text{Ch} = \frac{1}{4\pi} \int \frac{\mathbf{d}_p \cdot (\partial_x \mathbf{d}_p \times \partial_y \mathbf{d}_p)}{|\mathbf{d}_p|^3} d^2 p. \quad (1)$$

Here, $\partial_i = \partial/\partial p_i$ and integral is taken everywhere over the region of the momentum definition.

Proportionality of the quantized Hall conductance to the topological Chern number means that QAHE must be protected against perturbation due to disorder that is inevitably present in any real system. But the expression (1) is obtained for the case of the absence of disorder.



The authors of the work [12] showed that the quantized value of Hall conductance $\sigma_0 \text{Ch}$ (1) is robust against the perturbation due to cellular disorder in the crystal lattice. But results of this work do not show how the width of the QAHE plateau changes with increasing of disorder, how the Hall conductance depends on the parameters of the considered system beyond this plateau etc. To answer these questions, it is necessary to find an explicit expression for the intrinsic Hall conductance of a disordered topological insulator that is valid both in the QAHE regime and beyond. The purpose of article is the solution of this problem for the simplest model of a disordered topological insulator with broken time reversal symmetry, i.e., Chern insulator.

2. Model

Let consider the minimal model of the two-dimensional (2D, ||OXY), disordered Chern insulator whose one-particle Hamiltonian has the following form [13]

$$\mathcal{H} + U = \mathbf{d}_p \cdot \boldsymbol{\sigma} + U = \alpha(\mathbf{p} \cdot \boldsymbol{\sigma}) + M\sigma_z + U(\mathbf{r}). \quad (2)$$

Here, $\mathbf{p} = (p_x, p_y, 0)$ is the two-dimensional momentum, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector formed by the Pauli spin matrices, α is a constant of the spin-orbit coupling, $M(> 0)$ is the mean-field exchange energy splitting, and $U(\mathbf{r})$ is a white-noise Gaussian random potential field.

The energy eigenvalues in the clean ($U = 0$) system (2) are $\mathcal{E}_s(\mathbf{p}) = s|\mathbf{d}_p| = s\sqrt{M^2 + \alpha^2 \mathbf{p}^2}$ ($s = \pm 1$), $\mathbf{d}_p = (\alpha p_x, \alpha p_y, M)$. The positive ($s = 1$) and negative ($s = -1$) energy branches $\mathcal{E}_s(\mathbf{p})$ play the role of conduction and valence bands that are separated by gap $\mathcal{E}_g = 2M$. The quantum number $s = \pm 1$ is the eigenvalue of the operator $(\mathbf{d}_p \cdot \boldsymbol{\sigma})/|\mathbf{d}_p|$, that is an analog of the helicity operator $(\mathbf{p} \cdot \boldsymbol{\sigma})/|\mathbf{p}|$ and approaches it as $M \rightarrow 0$. As a consequence, the electrons in the states with energies $\mathcal{E}_s(\mathbf{p})$ have the spin projection $\pm \hbar/2$ onto the direction of vector \mathbf{d}_p .

We are interested here in intrinsic part of the anomalous Hall conductance. According to the Kubo-Středa formalism [14] it consists of two terms ($T = 0$)

$$\sigma_{xy}^{\text{int}} = \sigma_{xy}^{\text{I}} + \sigma_{xy}^{\text{II}} = \frac{\hbar e^2}{2\pi} \text{Tr} V_x G^R V_y G^A - |e|c \left(\frac{\partial n}{\partial B} \right)_{\mathcal{E}_F}. \quad (3)$$

Here, symbol $\text{Tr} = \text{tr Sp}$ denotes the trace both in spatial (Sp) and spin (tr) degrees of freedom, $V_i = \alpha \sigma_i$ are the Cartesian components of the velocity operator, and $G^{R(A)}$ is the averaged operator of one-particle retarded (R) or advanced (A) Green function (GF).

The second term, σ_{xy}^{II} , of conductance (3) is proportional to the derivative of electron concentration n with respect to magnetic induction B at a fixed value of the Fermi level \mathcal{E}_F . The expression (3) is written for the case of $B \neq 0$, but it remains valid in the limit $B \rightarrow 0$. Therefore, it can be used as starting point for calculation of the intrinsic anomalous Hall conductance. We adapt the model (2) for this purpose including an external transverse magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ by means of substitution $\mathbf{p} \mapsto \boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}/c$. We do not include here Zeeman energy as it produces no effect on the anomalous Hall conductance in the absence of a magnetic field.

3. Green function and densities of states

The averaged retarded (advanced) one-particle GF can be written in the Dyson form

$$G^{R(A)}(\mathcal{E}) = \frac{1}{\mathcal{E} - \mathcal{H} - \Sigma^{R(A)}(\mathcal{E})}, \quad (4)$$

where $\Sigma^{R(A)}(\mathcal{E})$ is the electron self-energy operator. Further we shall restrict ourselves to the self-consistent Born approximation (SCBA) for description of the one-electron states in the system under consideration (2). In this approximation the electron self-energy operator is determined by

equation $\Sigma^{R(A)}(\mathcal{E}) = W \text{Sp} G^{R(A)}(\mathcal{E})$, where W is the amplitude of the two-point correlator of the random field $U(\mathbf{r})$. Below we drop superscripts $R(A)$, if this does not lead to misunderstandings.

It is easy to verify, that Σ is diagonal in spin space and has the following matrix structure

$$\Sigma = \Sigma_e + \Sigma_m \sigma_z, \quad \Sigma_e = \frac{W}{2} \text{Tr} G, \quad \Sigma_m = \frac{W}{2} \text{Tr} \sigma_z G. \quad (5)$$

It follows that the averaged GF in SCBA (4) can be obtained from the one-particle GF of the clean system

$$G(\mathcal{E}) = \frac{1}{\mathcal{E} - \mathcal{H}} = \frac{\mathcal{E} + M\sigma_z + \alpha(\boldsymbol{\pi} \cdot \boldsymbol{\sigma})}{\mathcal{E}^2 - M^2 - 2m\alpha^2 \mathcal{H}_0} \quad (6)$$

with the help of substitutions $\mathcal{E} \mapsto \mathcal{E} - \Sigma_e$ and $M \mapsto M + \Sigma_m$. Here

$$\mathcal{H}_0 = \frac{1}{2m}(\boldsymbol{\pi} \cdot \boldsymbol{\sigma})^2 = \frac{\pi^2}{2m} + \frac{\hbar\omega_c}{2}\sigma_z \quad (7)$$

is the Hamiltonian of a free electron with ideal value of Zeeman coupling ($g = 2$) in an orthogonal magnetic field, $\omega_c = |e|B/mc$ is the cyclotron frequency. Further we will write the expressions for averaged GFs in the form (6) implying that the self-energies Σ_e and Σ_m are included in \mathcal{E} and M respectively. To put it differently, Σ_e and Σ_m determine the perturbation by a random field U of the one-electron energy levels and mean-field exchange energy splitting respectively.

Thus, (5) and (6) form a system of the self-consistent transcendental equations in the Σ_e and Σ_m . In the case $B = 0$, both of these self-energies are expressed in terms of unique function Φ

$$\Sigma_e = \mathcal{E}\gamma_0\Phi, \quad \Sigma_m = M\gamma_0\Phi, \quad |\gamma_0\Phi| \ll 1. \quad (8)$$

Here $\gamma_0 = W\mathcal{N}_F/2m\alpha^2$ is the dimensionless parameter of disorder, $\mathcal{N}_F = m/2\pi\hbar^2$ is the density of states (DOS) of two-dimensional free spinless electrons in the absence of a magnetic field, and the function Φ satisfies the self-consistent equation

$$\Phi = \ln \left[(1 + \gamma_0\Phi)^2 - (1 - \gamma_0\Phi)^2 \frac{\mathcal{E}^2}{M^2} \right]. \quad (9)$$

The numerical solution of this equation allows us to calculate the self-energies Σ_e and Σ_m and such characteristics of the spectrum of electron states as the total DOS $\mathcal{N}(\mathcal{E})$ and the difference of partial DOSs $\mathcal{N}_m(\mathcal{E})$ with opposite values of spin projections onto the OZ-axis (spin DOS)

$$\mathcal{N}(\mathcal{E}) = \mp \text{Im} \frac{1}{\pi} \text{Tr} G^{R(A)}(\mathcal{E}), \quad \mathcal{N}_m(\mathcal{E}) = \mp \text{Im} \frac{1}{\pi} \text{Tr} \sigma_z G^{R(A)}(\mathcal{E}). \quad (10)$$

The results of this calculation at the different values of the disorder parameter γ_0 are shown in the figure 1 (left and mid panels). As can be seen from this figure, the energy gap between valence and conductivity bands narrows with increasing of γ_0 .

4. Intrinsic anomalous Hall conductance

The contribution of σ_{xy}^I to the anomalous Hall conductance is calculated by direct substitution of (6), (7) into the first term of (3). In the case $B = 0$, it gives the following result

$$\sigma_{xy}^I = -\frac{\sigma_0}{2\pi} \frac{2\mathcal{E}M \text{Im} \Phi}{\mathcal{E}^2 + M^2 - (\mathcal{E}^2 - M^2)\gamma_0 \text{Re} \Phi}. \quad (11)$$

According to (8), this part of σ_{xy} becomes identical zero even in the presence of disorder, if the Fermi level lies within the energy gap (see figure 1).

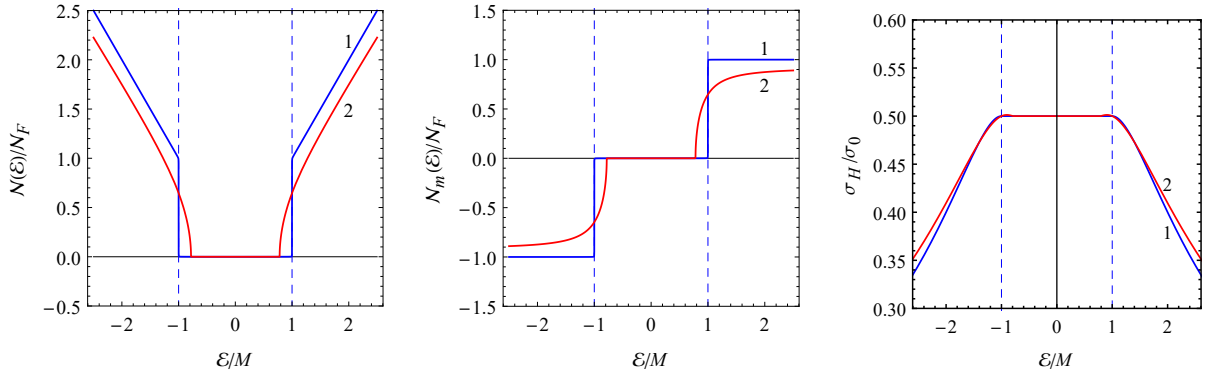


Figure 1. (color online) (Left panel) The energy dependence of the total DOS of the model (2) calculated for the values of disorder parameter $\gamma_0 = 0$ (line 1 (blue)) and $\gamma_0 = 0.04$ (line 2 (red)); (Mid panel) The same for the spin DOS; (Right panel) The same for the total σ_{xy} (3). The vertical thin dashed lines depict the positions of the valence and conductivity bands edges in the absence of disorder.

Thus, the nontrivial quantized Hall conductance should be determined entirely by Středa term σ_{xy}^{II} . Let consider more specifically its calculation. For this purpose, we have to find the explicit expression for the thermodynamic derivative $(\partial n / \partial B)_{\mathcal{E}_F}$ in the presence of an orthogonal magnetic field [see eq. (3)] and then pass to the limit $B \rightarrow 0$. By definition, $n = \int_{-\infty}^{\mathcal{E}_F} \mathcal{N}(\mathcal{E}) d\mathcal{E}$, therefore, the problem is reduced to the search for such expression for $\partial \mathcal{N}(\mathcal{E}) / \partial B$ in the presence of an orthogonal magnetic field that can be explicitly integrated over the energy \mathcal{E} . By direct differentiation of $\text{Tr } G$ with respect to the magnetic field induction B we obtain

$$\frac{\partial \text{Tr } G}{\partial B} = \frac{1}{B} \left[\left(1 - \frac{\partial \Sigma_e}{\partial \mathcal{E}} \right) \text{Tr } G - \frac{\partial \Sigma_m}{\partial \mathcal{E}} \text{Tr } \sigma_z G \right] - \frac{\partial}{\partial \mathcal{E}} \text{Tr } \frac{\partial \mathcal{H}}{\partial B} G. \quad (12)$$

Let consider the first term. The spectrum of Hamiltonian \mathcal{H}_0 (7) consists of two sets of equidistant Landau levels $\mathcal{E}_{\pm, n} = \hbar \omega_c (n + 1/2 \pm 1/2)$, where $n = 0, 1, 2, \dots$. Therefore, the traces $\text{Tr } G$ and $\text{Tr } \sigma_z G$ can be written as

$$\text{Tr } G = \mathcal{N}_F \left[\frac{\mathcal{E}}{m\alpha^2} \Phi(\mathcal{E}, M) - \frac{\hbar \omega_c}{\mathcal{E} - M} \right], \quad \text{Tr } \sigma_z G = \mathcal{N}_F \left[\frac{M}{m\alpha^2} \Phi(\mathcal{E}, M) - \frac{\hbar \omega_c}{\mathcal{E} - M} \right], \quad (13)$$

where

$$\Phi(\mathcal{E}, M) = \psi \left(\frac{M^2 - \mathcal{E}^2}{2m\alpha^2 \hbar \omega_c} \right) + \ln \frac{2m\alpha^2 \hbar \omega_c}{M^2}. \quad (14)$$

Here $\psi(z) = d/dz \ln \Gamma(z)$ is the digamma-function [15]. The second term in the right hand side cancels the divergence of the digamma-function as $B \rightarrow 0$ and provides the correct passage of (14) to (9) in this limit.

The direct substitution of (13), (14) into (12) allows us to find the analytic expression for the indefinite integral of the first term of (12). Similarly, we can calculate the second term in the right hand side of (12) that is already the derivative of a some function with respect to \mathcal{E} . Finally, taking into account the asymptotic behaviour of the gamma- and digamma-functions for large values of their arguments [15], we obtain the expression for σ_{xy}^{II} (3) in the limit $B \rightarrow 0$

$$\sigma_{xy}^{\text{II}} = \frac{\sigma_0}{2} \text{Im} \frac{1}{\pi} \ln \frac{\mathcal{E} - M}{\mathcal{E} + M}. \quad (15)$$

The substitutions $\mathcal{E} \mapsto \mathcal{E} - \Sigma_e^A$ and $M \mapsto M + \Sigma_m^A$ are assumed to be performed in (15). The results of numerical analysis of the total intrinsic Hall conductance $\sigma_{xy} = \sigma_{xy}^I + \sigma_{xy}^{II}$ [see eqs. (11) and (15)] at the different values of the disorder parameter γ_0 are shown in the figure 1 (right panel). As can be seen from this figure, the quantization of the anomalous Hall conductance is protected against perturbation due to disorder, if the Fermi level lies within the energy gap.

5. Conclusion

In the present work, the effect of disorder on the spectrum of one-particle states and on the behavior of the anomalous Hall conductance σ_{xy} in the simplest model of a Chern insulator is investigated in SCBA. The analysis of obtained expressions shows that the width of gap between valence and conductivity bands decreases as disorder parameter increases. In turn, it leads to the narrowing the energy region in which the regime of QAHE is realized.

As can be seen from the figure 1, the disorder has a pronounced effect on the behavior of conductance (3) outside the energy gap. However, σ_{xy}^I is partially cancelled by the ladder series for σ_{xy} [16]. Thus, the final conclusion on the behavior of the Hall conductance outside the energy gap can be made after taking into account the contribution of the side-jump mechanism.

It should be noted that perturbances of σ_{xy}^I (11) and σ_{xy}^{II} (15) cancel each other with very high accuracy. In particular, the relative deviation of the sum of these terms its quantized value $e^2/4\pi\hbar$ is at least an order of magnitude smaller than the dimensionless disorder parameter γ_0 .

In the case in question, the magnitude of quantized anomalous Hall conductance is equal to $e^2/4\pi\hbar$. The point is that the Hamiltonian (2) describes so-called marginal model in that the corresponding Chern number (1) takes the half-integer value $\text{Ch} = 1/2$. Therefore, it is of interest to perform a similar analysis for a more realistic model of topological insulator. The another important challenge is to reveal connection between topological invariants of one-electron states and intrinsic anomalous Hall conductance of disordered topological insulator.

We note that the SCBA used in the paper is correct provided that $k_F l \gg 1$. This inequality is violated near the edges of the valence and conduction bands $|\mathcal{E}| \simeq M$. In this energy region, SCBA gives only qualitative description of the behavior of the DOSs and Hall conductance.

Acknowledgments

This work was carried out within the state assignment of Russian Ministry of Science and Higher Education (theme ”Spin” No. AAAA-A18-118020290104-2).

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